An FPTAS for Optimizing a Class of Low-Rank Functions Over a Polytope

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Outline

- Low rank functions: definition and examples
- Our framework: assumptions and results
- Techniques: Pareto-optimal front and approximate Pareto-optimal front
- The approximation scheme and applications

Low Rank Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is of rank k if

$$f(x) = g(a_1^T x, \ldots, a_k^T x),$$

where $a_1, \ldots, a_k$ are k linearly independent vectors.

Low rank: k fixed and $k \ll n$.

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The Optimization Problem

Problem

$$\begin{aligned} \min / \max \quad & f(x) = g(a_1^T x, \ldots, a_k^T x) \\ \text{s.t.} \quad & Cx \geq d \end{aligned}$$

For example, f can be

- $f(x) = (a_1^T x) \cdot (a_2^T x)$ (multiplicative)
- $f(x) = (a_1^T x) \cdot (a_2^T x) + (a_3^T x) \cdot (a_4^T x)$ (bi-linear)
- $f(x) = \frac{a_1^T x}{b_1^T x} + \frac{a_2^T x}{b_2^T x}$ (sum-of-fractions)

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NP-Hardness Results

The following problems are NP-hard (Matsui 1996):

Problem 1

$$\begin{aligned} \min \quad & x_1 x_2 \\ \text{s.t.} \quad & Cx \geq d \end{aligned}$$

Problem 2

$$\begin{aligned} \max \quad & \frac{1}{x_1} + \frac{1}{x_2} \\ \text{s.t.} \quad & Cx \geq d \end{aligned}$$

Challenges

- Bi-linear functions and sum-of-ratios functions are neither quasi-concave nor quasi-convex.
- Can have multiple local optima, so any global minimization algorithm must avoid getting stuck into the local optima.
- Results known mostly for minimizing quasi-concave functions of low rank over a polytope (e.g. Goyal and Ravi (2009), Kelner and Nikolova (2007), Porembski (2004))

Fully Polynomial Time Approximation Scheme (FPTAS)

Consider an instance π of a family of minimization problems:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X \end{array}$$

Let x^{OPT} be the optimal solution of the given instance.

Definition

An FPTAS is a family of algorithms A_ϵ, such that for any $\epsilon > 0$ and for all instances π of the problem, the algorithm A_ϵ

- returns a solution $\hat{x} \in X$ such that $f(\hat{x}) \leq (1+\epsilon) f(x^{OPT})$, and
- has running time polynomial in $|\pi|$ and $1/\epsilon$.

Our Result

FPTAS for the following optimization problem for a fixed k:

Problem

$$\begin{aligned} \min / \max \quad & f(x) = g(a_1^T x, \ldots, a_k^T x) \\ \text{s.t.} \quad & Cx \geq d \end{aligned}$$

Assumptions

- $g(y) \leq g(y')$ for all $y \leq y'$.
- $g(\lambda y) \leq \lambda^c g(y)$ for all $\lambda > 1$ and some constant c.
- $a_i^T x > 0$ for all $i = 1, \ldots, k$ over the given polytope.

Our Result (contd.)

Examples of functions satisfying the above conditions:

- Multiplicative forms: $f(x) = \prod_{i=1}^{k} (a_i^T x)$
- Bi-linear forms: $f(x) = \sum_{i=1}^{k} (a_i^T x) \cdot (b_i^T x)$

The monotonicity assumption can be relaxed:

- For example, the sum-of-ratios form: $f(x) = \sum_{i=1}^{k} \frac{a_i^T x}{b_i^T x}$

The Solution Approach

Problem π

$$\begin{aligned} \min \quad & f(x) = (a_1^T x) \cdot (a_2^T x) \\ \text{s.t.} \quad & Cx \geq d \end{aligned}$$

Solution

- Let $f_i(x) = a_i^T x$.
- Compute an *approximate Pareto-optimal* frontier of the functions f_i.
- Return the best solution from the approximate Pareto-optimal frontier.

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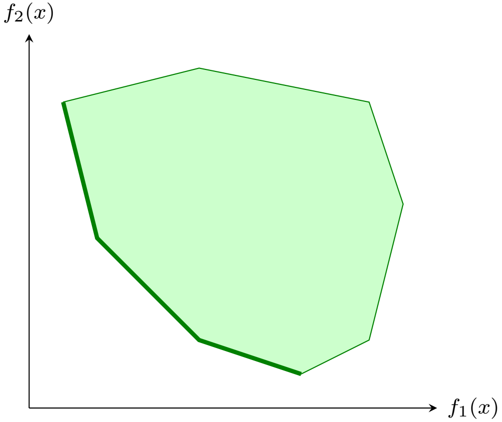
Pareto-optimal Frontier

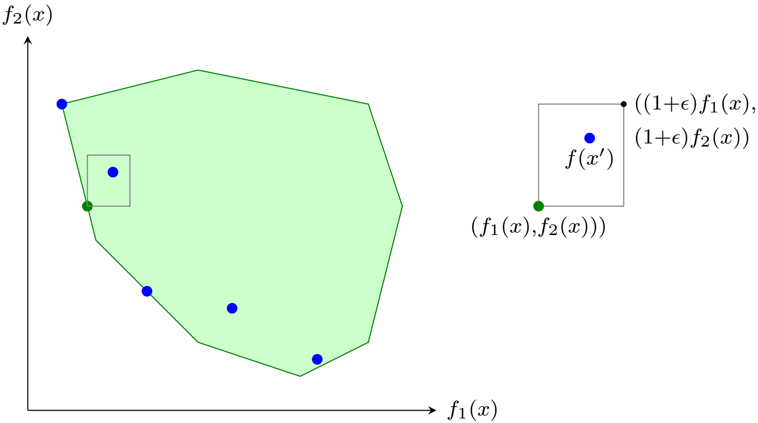
Pareto-optimal front $(P(\pi))$ is the set of all non-dominated solution points.



Approximate Pareto-optimal Frontier

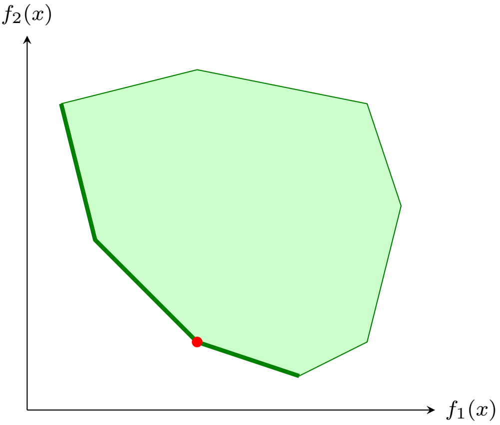
Set of solutions $P_\epsilon(\pi)$ such that:

for all feasible x, there is $x' \in P_\epsilon(\pi)$ such that $f_i(x') \leq (1+\epsilon) f_i(x)$.



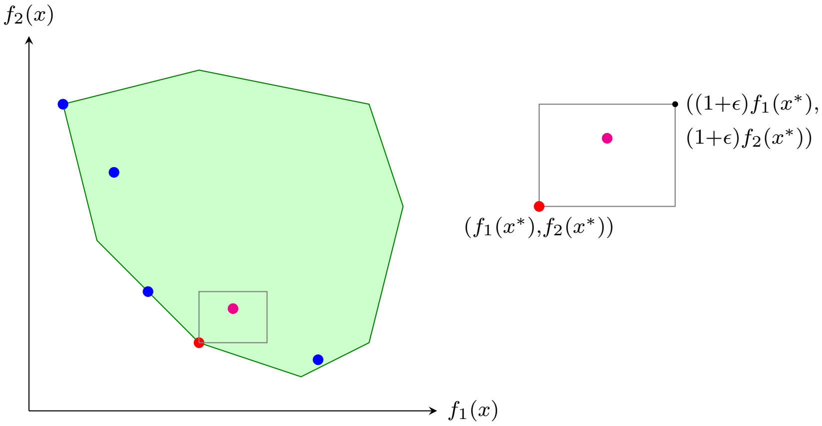
Lemma 1

An optimal solution of the problem π lies on the Pareto-optimal front.



Lemma 2

Let $\hat{x}$ be the solution in $P_\epsilon(\pi)$ that minimizes $f(x)$ over all the points in $P_\epsilon(\pi)$. Then $\hat{x}$ is a $(1+\epsilon)^2$-approximate solution.



The Gap Theorem (Papadimitriou and Yannakakis 2000)

For a fixed k, it is possible to find a $P_\epsilon(\pi)$ in time polynomial in $|\pi|$ and $1/\epsilon$ *iff* the following "gap problem" can be solved in polynomial time.

Gap problem

Given a k vector of values $(v_1, \ldots, v_k)$, either

(i) return a feasible x such that $f_i(x) \leq v_i$ for all $i = 1, \ldots, k$, or ..

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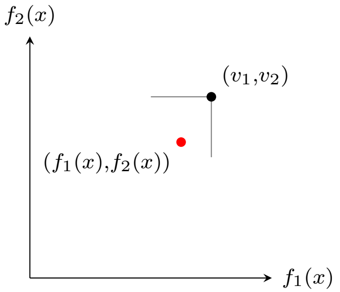

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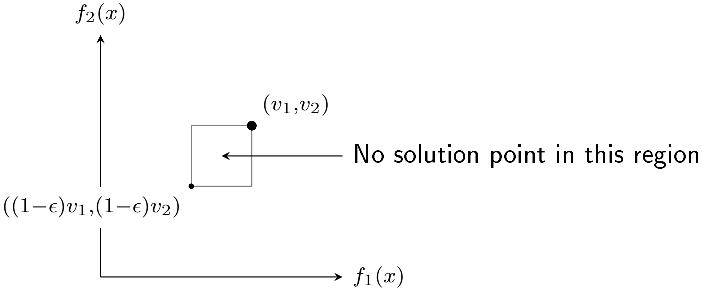
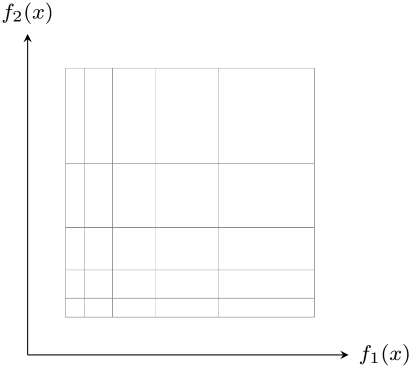
(ii) assert that there is no feasible x' such that $f_i(x') \leq (1-\epsilon)v_i$ for all $i = 1, \ldots, k$.



The Approximation Scheme

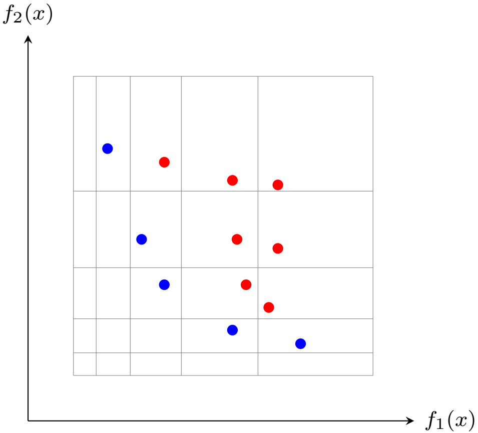
Divide the solution space into smaller hyper-rectangles, such that in each dimension, the ratio of successive divisions is equal to $1 + \epsilon'$.

(ϵ' depends on ϵ).



The Approximation Scheme

For each corner point, solve the gap problem. Return the set of undominated solution points.



Solving the Gap Problem

Same as checking the feasibility of the following LP, for each corner point $(v_1, \ldots, v_k)$:

Gap Problem LP

$$\begin{aligned} Cx &\geq d, \\ a_i^T x &\leq (1-\epsilon')v_i, \text{ for } i = 1, \ldots, k. \end{aligned}$$

Need to check feasibility of $\mathcal{O}\left(\left(\frac{\log(M/m)}{\epsilon}\right)^k\right)$ LPs.

Applications: Sum-of-Ratios Optimization

$$\min / \max f(x) = \frac{a_1^T x}{b_1^T x} + \ldots + \frac{a_k^T x}{b_k^T x}, \text{ s.t. } Cx \geq d.$$

- Application: Multi-stage shipping problem (Falk and Palocsay, 1992).
- Sum-of-fractions is not quasi-convex/quasi-concave in general, no approximation algorithms known.
- Our result: FPTAS when k is fixed.

Applications to Combinatorial Optimization Problems

- The framework can be applied to combinatorial optimization problems with similar objective functions as well.
- The gap theorem is still true, but needs to be solved differently.

Applications:

- Spanning tree/shortest path problem with a product objective function (probabilistic network design - Kern and Woeginger (2007)).
- Knapsack problem with a sum-of-ratios objective function.

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Summary

- We gave a general framework to optimize a class of low-rank functions over a polytope.
- The framework is applicable even when the function is not quasi-concave/quasi-convex.
- Can be used to get FPTAS for combinatorial optimization problems with non-linear objective functions as well.

Thank You!

Questions??